Infinite Series

Monotonic Sequence: If the sequence $\{a_n\}$ be such that either

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots$$

OR $$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$$

then it is said to be monotonic.

In the first case, in which each element is equal to or greater than the preceding element i.e $a_n \leq a_{n+1}$ $\forall n$, the sequence is said to be monotonic increasing.

In the 2nd case in which each element is equal to or less than the preceding element i.e $a_n \geq a_{n+1}$ $\forall n$, the sequence is said to be monotonic decreasing. (m.d)

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Ratio test **Th** If $\{a_n\}$ be a sequence such that $\lim \frac{a_{n+1}}{a_n} = l$ where $|l| < 1$, then

$$\lim_{n \to \infty} a_n = 0$$

Proof Since $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = l$ therefore corresponding to an arbitary positive quantity ε, an integer m can be found such that

$$\left| \frac{a_{n+1}}{a_n} - l \right| < \varepsilon \text{ for } n \geq m$$

$$\left| \frac{a_{n+1}}{a_n} \right| < |l| + \varepsilon \text{ for } n \geq m$$

Again, since $|l| < 1$, we can find k such that $|l| + \varepsilon = k < 1$

Thus $\left| \frac{a_{n+1}}{a_n} \right| < k$ for $n \geq m$

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Hence putting $n = m, m+1, m+2 \ldots$ we have

$$\left|\frac{a_{m+1}}{a_m}\right| < K, \left|\frac{a_{m+2}}{a_{m+1}}\right| < K$$

$$\left|\frac{a_{m+3}}{a_{m+2}}\right| < K \ldots\ldots \left|\frac{a_n}{a_{n+1}}\right| < K$$

Multiplying the above ratios, we get

$$\left|\frac{a_n}{a_m}\right| < K^{n-m} \Rightarrow |a_n| < K^n \left|\frac{a_m}{K^m}\right|$$

$$\Rightarrow |a_n| < cK^n \text{ where } c = \left|\frac{a_m}{K^m}\right|$$

Now take the limit as $n \to \infty$

$\therefore$ c is finite and $K^n \to 0$ as $n \to \infty$ for k is a fixed number < 1, therefore

from (1) ; we get $\lim\limits_{n \to \infty} a_n = 0$